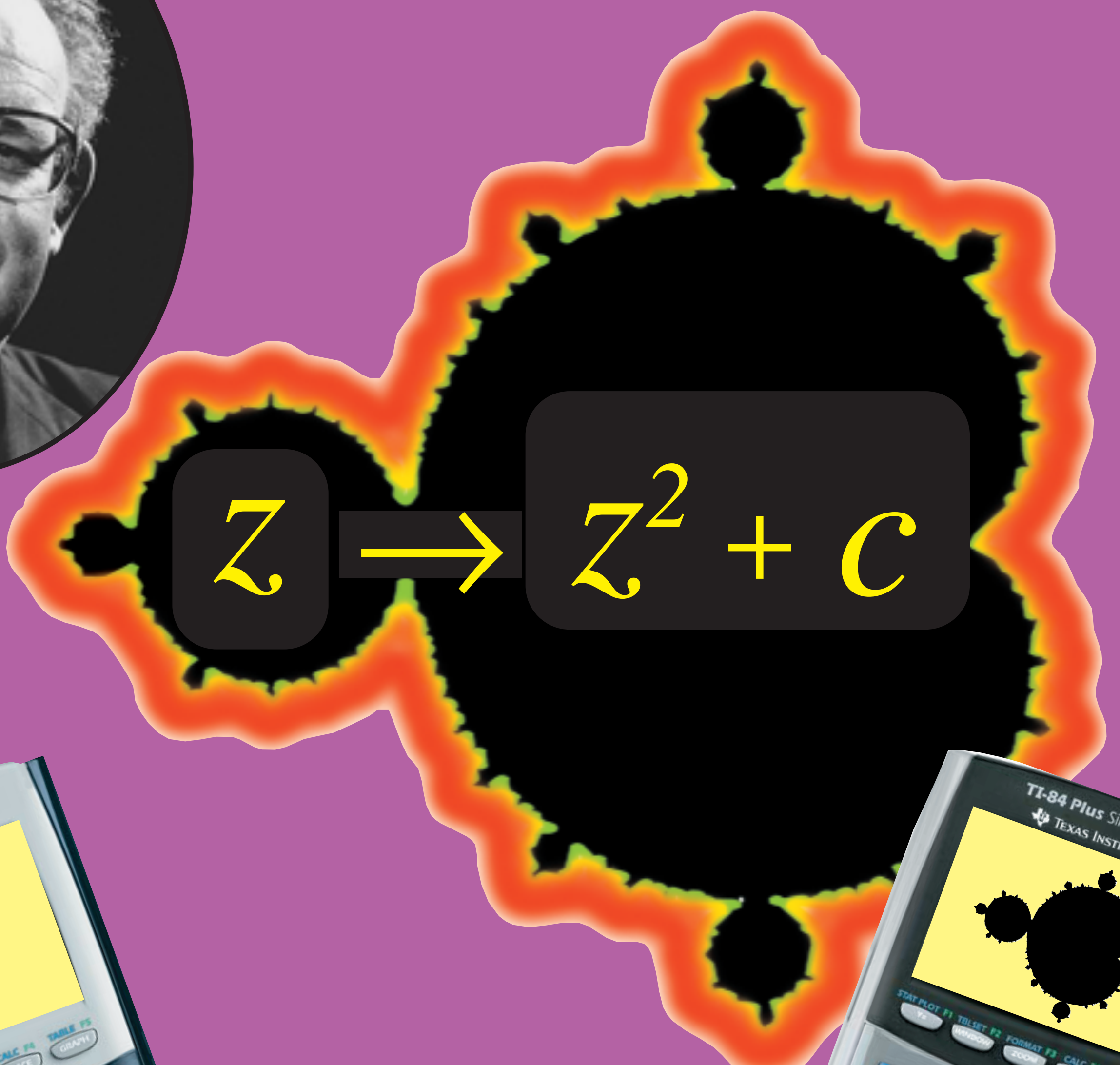
Advanced Algebra

With the TI-84 Plus Calculator

Benoit Mandelbrot



Brendan Kelly

Exploration 9 The Divine Proportion & the Golden Ratio

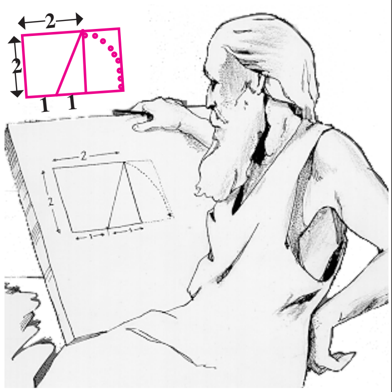
Leonardo da Vinci
1452–1519

> The merit of painting lies in the exactness of reproduction. Painting is a science and all sciences are based on mathematics. No human inquiry can be a science unless it pursues its path through mathematical exposition and demonstration.
>
> – Leonardo da Vinci

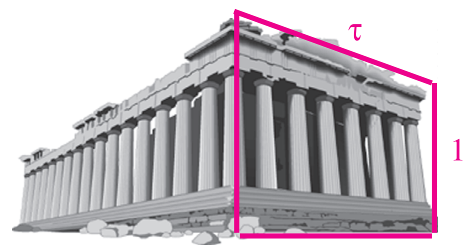
Mona Lisa
c. 1503–05

Over 2000 years ago, the ancient Greeks studied the esthetics of proportion. They sought the dimensions of a rectangle that could be partitioned into a square and a rectangle of the same shape as the original. A rectangle with such a shape was deemed to be a *golden rectangle*.

The diagram shows how they constructed a golden rectangle from a 2×2 square using a pair of compasses.



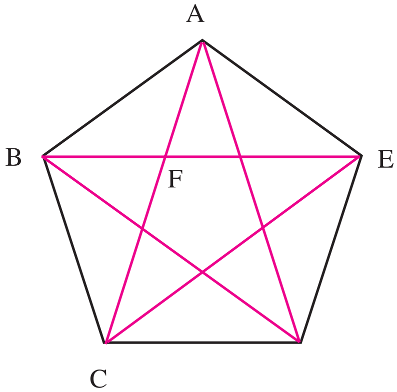
The Greeks used the golden rectangle in their architecture. When the Parthenon was completed in 438 B.C., its front elevation displayed the shape of a golden rectangle. That is, its width was about 1.618 times its height. This golden ratio or "golden section" was represented by τ, the first letter in the ancient Greek word τομη meaning "the section."



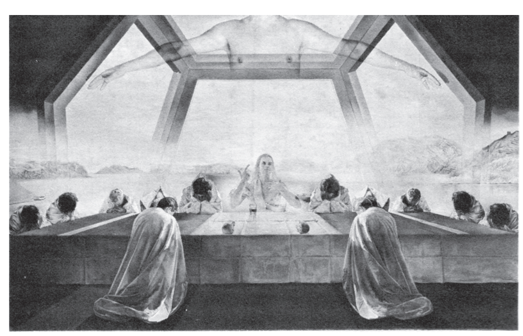
In 1509, mathematician Luca Pacioli published his *De Divina Proportione* that was illustrated by Leonardo da Vinci. In this treatise, he extolled with enthusiasm the special properties of τ that he called "the divine proportion." In this treatise, he stated that the ninth property of τ, "the best of all" is that two crossing diagonals of a regular pentagon divide one another in the golden ratio, τ:1.

For example:

$$\frac{BE}{FE} = \frac{FE}{BF} = \tau$$



During the Renaissance, mathematicians became enamored with the properties of τ. Its ubiquity in all fields of human endeavor prompted some scholars to regard its properties as mystical manifestations of God's grand plan. Fascination with τ has continued throughout the centuries. Salvador Dali's painting *The Sacrament of the Last Supper* is a golden rectangle in shape and features in the background a huge dodecadhedron with pentagonal faces.

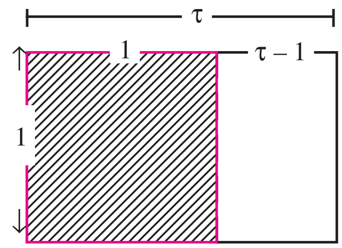


Worked Examples

Worked Example 1

When a square is removed from a golden rectangle, the part remaining is a rectangle of the same shape as the original. Calculate τ, the length-to-width ratio of a golden rectangle.

Solution



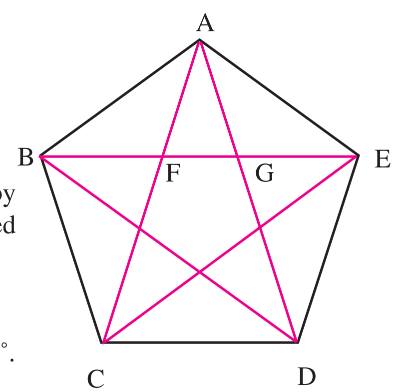
If we let τ denote the length of the original rectangle, then its length-to-width ratio is τ:1. When the unit square is removed from it, the remaining rectangle has length-to-width ratio 1:(τ – 1). The requirement that both rectangles have the same shape, i.e., length-to-width ratio is given by: $\frac{\tau}{1} = \frac{1}{\tau - 1}$

Cross multiplication yields $\tau^2 - \tau - 1 = 0$. This is a quadratic equation with roots: $\frac{1 \pm \sqrt{5}}{2}$.
We denote the positive root by τ and the negative root by σ. Then,

$\tau = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{1-\sqrt{5}}{2}$ The decimal expansion of τ is 1.618033989…

Worked Example 2

Prove that intersecting diagonals of a regular pentagon divide each other in the golden ratio. For example, FE:BF = BE:FE.
Deduce that the diagonal of a regular pentagon is τ times its side length.



Solution

Since all sides of a regular pentagon are of equal length, and all the angles formed by adjacent sides are 108˚, it follows from SAS (side-angle-side) that all triangles formed from two sides of the pentagon are congruent. Since ΔABC is congruent to ΔABE, $\angle BAF = \angle AEB$. Since, all these triangles are isosceles, $\angle ABE = \angle AEB$.
Therefore, $\angle ABE = \angle AEB = \angle BAF$.
Since $\angle BAE = 108^\circ$, then $\angle ABE = \angle AEB = \angle BAF = 36^\circ$. Therefore $\angle CAD = 36^\circ$.
Since Δ ABF and Δ ABE are 36˚–36˚–108˚ triangles, they are similar. Therefore,

$\frac{BA}{BF} = \frac{BE}{BA}$ Since ΔAFE is isoceles, FE = AE = BA, so $\frac{FE}{BF} = \frac{BE}{FE}$. ❶

If the pentagon has sides of unit length, i.e., AE = FE = 1 and BE is of length x units, then ❶ implies: $\frac{1}{x-1} = x$. This equation was shown in *Worked Example 1* to have root τ. Hence the diagonal BE is τ times the length of the side of the pentagon.

Worked Example 3

a) Evaluate $\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$

b) Enter a positive number and press ENTER 2nd [√] 2nd [ANS] + 1 ENTER.
What happens when you press ENTER repeatedly?

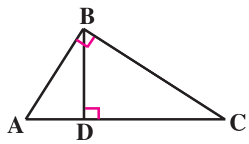
Solution

a) Let $x = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$. Then $x = \sqrt{1+x}$. Squaring both sides yields $x^2 = 1 + x$, or $x^2 - x + 1 = 0$. That is, $x = \tau$, the golden ratio.

b) When we take any positive number, add 1 and then take the square root of the result, and repeat this process, we are performing the operations in Part a), so our iteration eventually converges to τ.

Exercises & Investigations

❶. An altitude BD is dropped from vertex B to side AC of right triangle ABC. According to the *Mean Proportional Theorem*, AD:BD = BD:AC. If BD is one unit in length, what is the shortest possible length of AC?



❷. a) Prove that τ is the negative reciprocal of σ.
b) How could you have proved Part a) using the only the fact that τ and σ are roots of the equation $\tau^2 - \tau - 1 = 0$?

❸. Evaluate:

a) $\dfrac{\tau^3 + \tau^{-3}}{\sqrt{5}}$ b) $\tau^3 - \tau^{-3}$ c) $\dfrac{\tau^6 - \tau^{-6}}{\sqrt{5}}$

❹. Using the method in *Worked Example* 3, evaluate the following infinite continued fraction

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$$

where 1 + … means the fraction continues ad infinitum.

❺. Enter a positive number and press:

x^{-1} + 1 ENTER

Without entering any other number press that sequence of keys again. Repeat this 10 times. What do you discover?

❻.Using the diagram in *Worked Example* 2 and the sine and/or cosine laws:
a) List three non-similar triangles, giving their angles and corresponding side-lengths. (Assume AB has length 1 unit.)

b) Use the results of Part a) to compute sin 36˚ and cos 36˚.

c) What is the ratio of the area of pentagon ABCDE to the area of the small inverted pentagon with side FG?

❼.a) Use your calculator to create a table that displays the values of $\tau/\sqrt{5}$, $\tau^2/\sqrt{5}$, $\tau^3/\sqrt{5}$, …$\tau^n/\sqrt{5}$. What pattern do you notice in this sequence?

b) Use your calculator to create a table that displays the terms of the sequence: $(-\tau)^{-2}/\sqrt{5}$, $(-\tau)^{-3}/\sqrt{5}$, …$(-\tau)^{-n}/\sqrt{5}$.

c) Using Parts a) and b) display the terms of the sequence: $\tau^n/\sqrt{5} - (-\tau)^{-n}/\sqrt{5}$. Explain what you discover.

d) Use your result in Part c) to write a formula for f_n in terms of τ where f_n denotes the n^{th} term of the Fibonacci sequence. Use your formula to prove that as n becomes very large, f_{n+1}/f_n approaches τ.

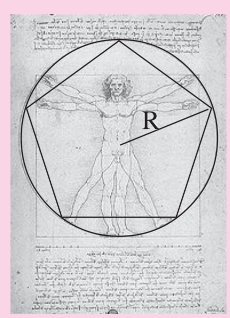
The DaVinci Code — The Mysterious ϕ

In 2006, the movie *The DaVinci Code* was released and it became a blockbuster success. Based on the best-selling novel by Dan Brown, it popularized the special properties of τ (now called Phi and denoted ϕ) the second most famous number in mathematics. (Of course, you know the most famous number.) It also popularized the Fibonacci sequence that you met in *Exploration 1*.

In *The DaVinci Code*, protagonist, Robert Langdon instructs his college students: *Nobody understood better than Da Vinci the divine structure of the human body…He was the first to show the the human body is literally made of building blocks whose proportional ratios always equal PHI…Measure the distance from the tip of your head to the floor. Then divide that by the distance from your belly button to the floor. Guess what number you get…Yes PHI.*

Langdon goes on to assert that ϕ is also the quotient obtained by dividing the distance from the shoulder to the finger tips by the distance from the elbow to the findertips.. It is also the quotient obtained by dividing the distance from the hip to the floor by the distance from the knee to the floor.

To illustrate the prominence of ϕ in the proportions of the human body, Leonardo da Vinci created in 1513 his famous sketch *Vitruvian Man*.



❽.a)What is the ratio of the height of *Vitruvian man* to his armspan?

b) Prove that the ratio of the length of a side of the regular pentagon inscribed in a circle to the radius of that circle is:

$$\sqrt[4]{5} : \phi^{\frac{1}{2}}$$

c) Assuming that the center of the circle is at the navel and that Robert Langdon's statements are correct, show that the ratio of the height of the vitruvian man to the length of a side of the regular inscribed pentagon is given by:

$$\phi^{\frac{3}{2}} : \sqrt[4]{5}$$