Algebra 2 *with* TI-nspire

Semester 1

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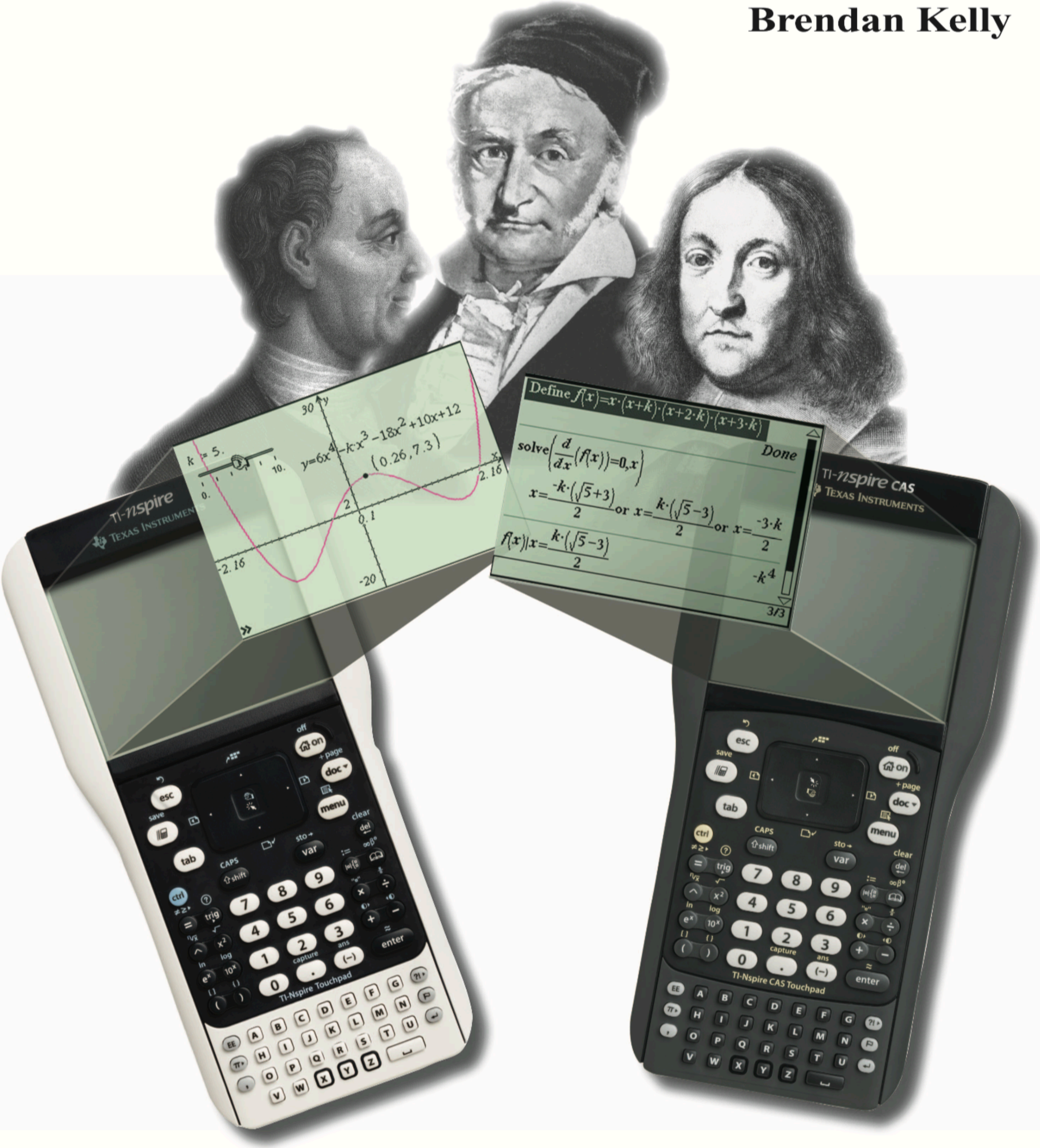


Table of Contents

Table of Contents

Unit 3 Matrices

Unit 4 Linear Systems

Unit 5 Quadratic Functions & Equations

Table of Contents

Unit 6 Polynomials & Polynomial Equations

Exploration 8: Using Formulas to Define Sequences

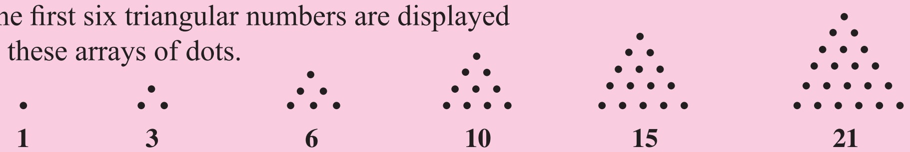
The Pythagoreans Celebrate the Sunrise– painting by Fyodor Bronnikov

Believing that the "personalities" of numbers determine the character of the objects they describe, the Pythagoreans of ancient Greece (see *Exploration 3*) associated numbers with shapes.

Numbers that can be displayed in n rows of n dots were called *square numbers*. Numbers that can be represented by triangular arrays of dots (such that the n^{th} row contains n dots) were called *triangular numbers*, and so on.

Example 1

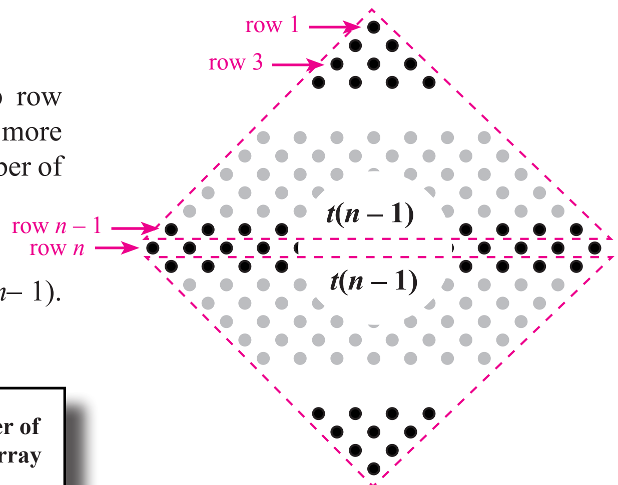
The first six triangular numbers are displayed in these arrays of dots.



a) Find a formula for the nth triangular number $t(n)$ as a function of n, where n is a positive integer.

b) Use this formula to calculate the 50th triangular number.

Solution



a) The diagram shows an $n \times n$ array of dots, i.e., n^2 dots.

The top triangle contains $n - 1$ rows of dots with the top row containing a single dot and each successive row containing one more dot, down to the bottom row that contains $n - 1$ dots. The number of dots in the top triangle is the $n - 1^{st}$ triangular number $t(n - 1)$.

Similarly, the number of dots in the bottom triangle is $t(n - 1)$. The total number of dots in the two triangles is therefore, $2t(n - 1)$. Since there are n dots in row n, we write:

The number of dots in the two triangles	+	The number of dots in row n	=	Total number of dots in the array
$2t(n-1)$	+	n	=	n^2

Subtracting n from both sides of this equation and dividing by 2 yields $t(n-1) = \dfrac{n^2 - n}{2}$.

Replacing $n - 1$ by n in this formula and simplifying, we obtain $t(n) = \dfrac{n^2 + n}{2}$.

b) Substituting $n = 50$ into the formula, we obtain $t(50) = (50^2 + 50)/2$ or 1275.
That is, the 50th triangular number is 1275.

When the triangular numbers are arranged as an ordered list $\{t(1), t(2), t(3), \ldots, t(n), \ldots\}$, they constitute a *sequence.*

Definition: Any function u defined for all positive integers is said to define a *sequence*, and $u(n)$ is used to denote the n^{th} *term* of the sequence.

Example 2

Write the first 10 terms of each sequence and evaluate its 20th term.

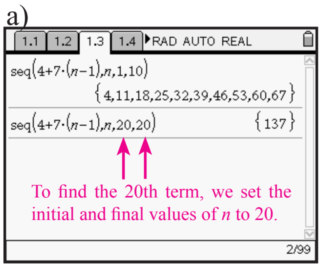
a) $u(n) = 4 + 7(n - 1)$ b) $u(n) = 2^n$ c) $u(n) = \left(\frac{3}{4}\right)^n$ d) $u(n) = 3n^2 + 2n - 1$

Solution

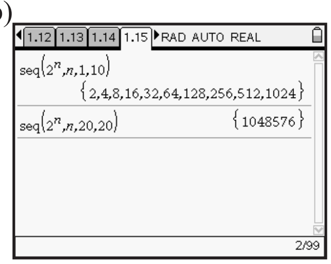
To find the first 10 terms of each sequence, we substitute values of n from 1 to 10 into each formula $u(n)$. For example, substituting $n = 1$ into $u(n) = 4 + 7(n - 1)$ yields $u(1) = 4 + 7(1 - 1)$ or 4, $u(2) = 4 + 7(2 - 1)$ or 11, $u(3) = 4 + 7(3 - 1)$ or 18, and so on… To avoid this tedium, we type the *seq(* command in the calculator application of TI-*n*spire as shown in the displays.

The syntax for the ***seq(*** command is **seq(**$u(n)$, n, initial value of n, final value of n**).** (formula → $u(n)$; variable → n)

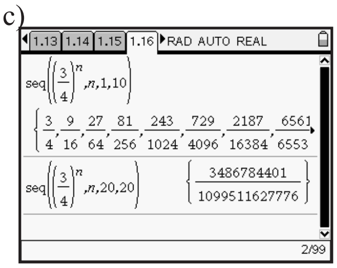
a)



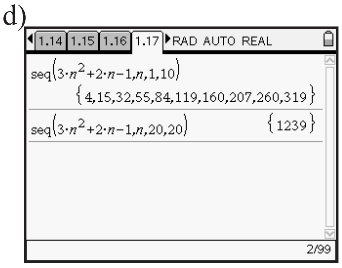
b)



c)



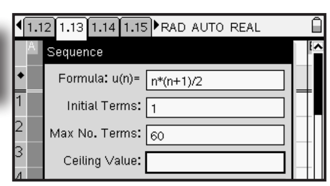
d)



The top lines in each display show the first 10 terms of each sequence.
The bottom lines in these displays show that the 20th terms are respectively:
137, 1,048,576, 3486784401/1099511627776 and 1239.

Example 3

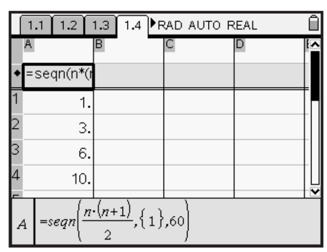
Use a spreadsheet to display the first 60 triangular numbers.



Solution

To access the *Lists & Spreadsheets* application we press: [home] [3].

We place the cursor in the header row of Column A and enter: [menu] [3] [1] to obtain the *sequence* template shown in the display. Since a sequence is a function defined on the integers, the template uses n rather than x as the variable and $u(n)$ rather than $f(x)$. Therefore, to complete the template, we enter the formula for the nth triangular number in the box for $u(n)$ as in the screen display. On pressing [enter], we obtain the triangular numbers in Column A. We can scroll down to see the rest.



Worked Examples

Some sequences have special properties that make them important in a variety of applications. Among these are the *arithmetic sequence* and the *geometric sequence.*

Arithmetic Sequences

Arithmetic sequences are special sequences in which the terms increase or decrease linearly. For example, successive terms of the sequence 4, 11, 18, 25, …in *Example 2a* differ by 7, so this is an arithmetic sequence.

7 7 7

Definition: Any sequence in which successive terms differ by the same constant is called an *arithmetic sequence*. The constant is called the *common difference*.
The first few terms of a general arithmetic sequence with first term a_1 and common difference d are given by:

$$a_1,\ a_1 + d,\ a_1 + 2d,\ \ldots\ \ a_1 + (n-1)d,\ \ldots$$

common difference d $\quad$ n^{th} term $u(n)$

Geometric Sequences

Geometric sequences are special sequences that can be produced by multiplying the first term a by a constant r to get the second term, and then multiplying the second term by the same constant r, and so on, so that each term is r times the previous term. For example, each term in the sequence 2, 4, 8, 16, … in *Example 2b* is double the previous term, so this is a geometric sequence with $r = 2$.

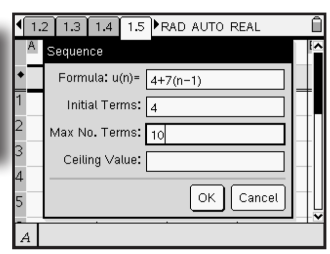
Definition: A sequence having successive terms in a fixed ratio is called a *geometric sequence*. The fixed ratio is called the *common ratio*. The first few terms of a general geometric sequence with first term, a and common ratio, r are given by:

$$a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1},\ \ldots$$

common ratio r $\quad$ n^{th} term $u(n)$

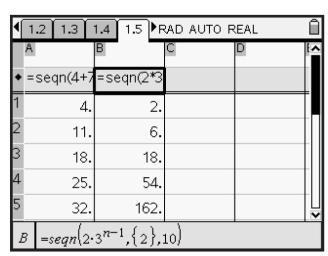
Example 4

Display on a spreadsheet the first 10 terms of:
a) an arithmetic sequence with first term 4 and common difference 7.
b) a geometric sequence with first term 2 and common ratio 3.



Solution

a) The arithmetic series has $a = 4$, $d = 7$, so $u(n) = a + (n–1)d$ or $4 + 7(n–1)$.
We proceed as in *Example 3* to enter $u(n) = 4 + 7(n–1)$ into the sequence template.
The completed template is shown in the top display.
On pressing **enter**, we obtain the first 10 terms of the arithmetic sequence in Column A.



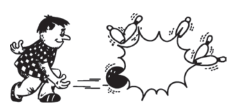
b) The geometric series has $a = 2$, $r = 3$, so $u(n) = ar^{n–1}$ or $2•3^{n–1}$.
We proceed as in *part a* to enter $u(n) = 2•3^{n–1}$ into the sequence template.
On pressing **enter**, we obtain the first 10 terms of the geometric sequence in Column B.

Exercises and Investigations

❶. Write a sentence to define each of the following terms. Give an example of each.
a) a finite sequence
b) an infinite sequence
c) an arithmetic sequence
d) a geometric sequence

❷. Indicate whether the following statement is true or false: *An infinite sequence is a function defined on the positive integers.* Justify your answer.

❸. The bowling pins in Super-Bowl-A-Rama are arranged in a triangle of 8 rows such that the first row contains one pin and each successive row contains one pin more than the previous row. How many pins are there?



❹. a) Calculate the 23rd triangular number.
b) Use your answer in *part a* to calculate the 24th triangular number.

❺. List the first 5 terms of each sequence without using technology.

a) $u(n) = 95 - 9(n - 1)$ b) $u(n) = 3^{n-1}$
c) $u(n) = 8 - 2^n$ d) $u(n) = 2^{-n}$
e) $u(n) = \dfrac{3n^2 - n}{2}$ f) $u(n) = \left(1 + \dfrac{2}{n}\right)^n$

Identify each sequence as an arithmetic sequence, geometric sequence, or neither of these.

❻. Use the *seq(* command as in *Example 2* to verify your answers in *Exercise* ❺.

❼. Display on a spreadsheet the first 10 terms of each of the sequences in *Exercise* ❺.

❽. Calculate $u(18)$ for each sequence in *Exercise* ❺.

❾. a) The first three terms of an arithmetic sequence $u(n)$, are: 17, 23, and 29. Write a formula for the nth term.

b) Calculate the 10th term of the sequence in *part a*.

❿. a) The first three terms of a geometric sequence $u(n)$, are: –5/3, 5/4, and –15/16. Write a formula for the nth term.

b) Calculate the 10th term of the sequence in *part a*.

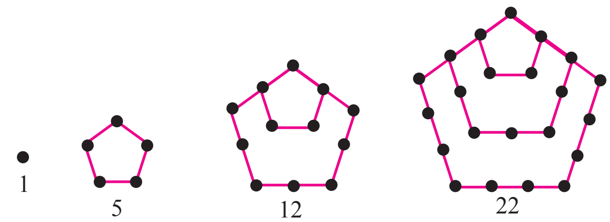
⓫. Find the 10th term of a geometric series with $u(5) = 48$, and $u(8) = 384$.

⓬. A particular geometric sequence has $u(3) = 8u(6)$. State the first term and the common ratio of the sequence if the 15th term is the reciprocal of the first term, i.e., $u(15) = 1/u(1)$.

⓭. a) Follow the procedure in *Example 3* to list in Column A of a spreadsheet the first 30 terms of the sequence defined by $u(n) = n^2 + 2n + 1$.

b) In the header row of Column B, enter **=√a[ ]**. Describe the numbers in Column B. Why are they all integers?

⓮.



The diagrams show the first 4 pentagonal numbers. The nth pentagonal number is given by $u(n) = an^2 + bn$ for some values of a and b. Find the values of a and b and then calculate the 15th pentagonal number $u(15)$.

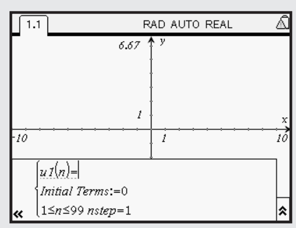
⓯. The sum of three consecutive integers in an arithmetic sequence is 48. The middle integer is also the middle of three consecutive integers in a geometric sequence. Find the three numbers in the geometric sequence if their sum is 84.

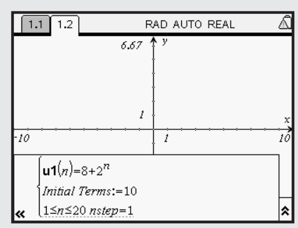
TI-nspire Investigation

Graphing Sequences

To graph the first 20 terms of the sequence $u(n) = 8 + 2^n$, we access *Graphs & Geometry* by pressing [home] [2].

Then we press [menu] [3] [5] to obtain the template for the sequence plot shown in the display below left.





We complete the template as in the display above right setting $u1(n) = 8 + 2^n$, the initial term to 6, and the number of terms 20. On pressing [enter], we see only three terms of the sequence. To view the 20 terms, we press [menu] [4] [A].

Use this procedure to graph the first 10 terms of each sequence. Trace along each graph to find $u(5)$. Which sequence has the largest growth rate?

a) $u(n) = 8 + 2^n$ b) $u(n) = 20 - n + 2n^2$
c) $u(n) = 30 + 3n$ d) $u(n) = 1 - n^2 + 2n^3$

Answers to the Exercises & Hints for the Investigations

Exploration 8

❶. a) A *finite sequence* is a function $u(n)$ defined on the set of positive integers from 1 to N, such that $u(n)$ is the n^{th} term of the sequence.
2, 4, 6, …30 is a finite sequence having 15 terms.

b) An *infinite sequence* is a function $u(n)$ defined on the set of all positive integers, such that $u(n)$ is the n^{th} term of the sequence.
The sequence 2, 4, 6, 8, …, $2n$, … is an infinite sequence.

c) An arithmetic sequence is any sequence for which the n^{th} term can be expressed in the form $u(n) = a + (n–1)d$ where a and d are constants. The first term is a and the common difference is d.
The sequence 5, 8, 11, …, $5 + 3(n – 1)$, … is an arithmetic sequence with $a = 5$ and $d = 3$.

d) A geometric sequence is any sequence for which the n^{th} term can be expressed in the form $u(n) = ar^{n–1}$ where a and r are constants. The first term is a and the common ratio is r.
The sequence 3, 6, 12, 24, … $32^{n – 1}$, … is a geometric series with $a = 3$ and $r = 2$.

❷. The statement is true by definition.

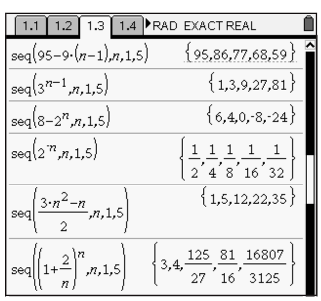
❸. The number of pins in successive rows are 1, 2, 3, … 8. This is the sequence of natural numbers from 1 to 8. The total number of pins is the 8th triangular number $t(8) = (8^2 + 8)/2$ or 36 pins.

❹. a) The *nth* triangular number is given by $t(n) = n(n + 1)/2$. Therefore the 23rd triangular number is $t(23) = 23(24)/2$ or 276.

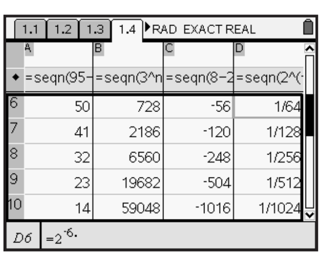
b) $t(23) = 1 + 2 + 3 + … + 23$ and $t(24) = t(23) + 24 = 276 + 24$ or 300.

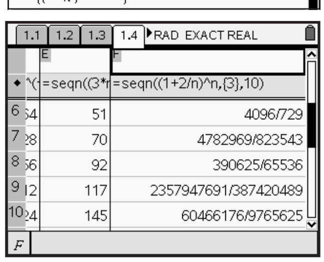
❺. a) arithmetic sequence {95, 86, 77, 68, 59}
b) geometric sequence {1, 3, 9, 27, 81}
c) neither type {6, 4, 0, –8, –24}
d) geometric sequence {1/2, 1/4, 1/8, 1/16, 1/32}
e) neither type {1, 5, 12, 22, 35}
f) neither type {3, 4, 125/27, 81/16, 16807/3125}

❻. The display verifies the answers in *Exercise* ❺.



❼. The displays below show the 6th through 10th terms of each sequence. The first five terms of each sequence are given in the display on the right.





❽. The 18th term of each sequence in *Exercise* ❺ is:
a) $u(18) = –58$ b) $u(18) = 1/387420489$ c) $u(18) = –262\ 136$
d) $u(18) = 1/(262\ 144)$ e) 477
f) $u(18) = 10^{18}/15009463529699912 \approx 6.662463…$

Exploration 8 cont'd

❾. a) The first term is $a = 17$ and the common difference is 23 – 17 or 6. The nth term is $u(n) = 17 + 6(n – 1)$.

b) $u(10) = 17 + 6(9) = 71$

❿. a) The first term is $a = –5/3$ and the common ratio is:
$r = (5/4)/(–5/3)$ or –3/4. The nth term is $u(n) = (–5/3)(–3/4)^{n – 1}$.

b) $u(10) = (–5/3)(–3/4)^9 = 32805/262144 \approx 0.125141…$

⓫. Since $u(n) = ar^{n – 1}$ and $u(5) = 48$, then $ar^4 = 48$. ①
Similarly, $ar^7 = 384$. ②
Dividing ② by ① yields $r^3 = 384 \div 48$ or 8. Therefore $r = 2$
Substitution of 2 for r in ① yields $a = 3$

⓬. Let $u(n) = ar^{n–1}$. Then $u(3) = 8u(6)$ implies $ar^2 = 8ar^5$. So, $r^3 = 1/8$, i.e. $r = 1/2$. Also, $u(15) = 1/u(1)$ so $ar^{14} = 1/a$.
Therefore, $a^2 = r^{-14}$, i.e., $a = \pm r^{-7} = \pm(1/2)^{-7}$ or ±128.
Therefore, $a = \pm128$ and $r = 1/2$.

⓭. b) Since $n^2 + 2n + 1 = (n + 1)^2$, then $u(n) = (n + 1)^2$ and $\sqrt{u(n)} = n + 1$. Therefore all the numbers in Column A are perfect squares and all the numbers in Column B are integers.

⓮. The diagrams show that $u(1) = 1$ and $u(2) = 5$. Substituting $n = 1$ and 2 into the formula for $u(n)$, we obtain $a + b = 1$ and $4a + 2b = 5$. Solving this pair of equations for a and b, we obtain $a = 3/2$ and $b = –1/2$.
Therefore $u(n) = (3/2)n^2 – (1/2)n$ or $n(3n – 1)/2$.

Substituting $n = 15$ into this formula yields $u(15) = 15(44)/2$ or 330. That is, the 15th pentagonal number is 330.

⓯. Denote the three integers in arithmetic sequence by $m – b$, m, and $m + b$. Since their sum is 48, we have $(m – b) + m + (m + b) = 48$ or $3m = 48$. Therefore $m = 16$.

Since the middle integer $m = 16$ is also the middle number of three other consecutive integers in a geometric sequence, there is a common ratio r such that the three consecutive integers can be written as $16/r$, 16, and $16r$. Since the sum of these integers is 84, $16/r + 16 + 16r = 84$. That is, $4/r + 4 + 4r = 21$. Multiplication of both sides of this equation by r yields $4r^2 – 17r + 4 = 0$. Solving this equation yields $r = 1/4$ or 4. The three numbers in the geometric sequence are 4, 16 and 64 or 64, 16 and 4.

Exploration 9

❶. The statement, *The sequence u(n) can be defined by recursion,* means that $u(n)$ can be defined in terms of one or more previous terms such as, $u(n–1)$ and/or $u(n–2)$.

❷. a) The nth term is the $(n – 1)$th term plus 7. This is an arithmetic sequence with first term 5 and common difference 7.

b) The nth term is double the $(n – 1)$th term. This is a geometric sequence with first term 11 and common ratio 2.

c) The nth term is the $(n – 1)$th term less 5. This is an arithmetic sequence with first term –3 and common difference –5.

d) This is a geometric sequence with first term 1 and common ratio ½.

❸. a) $u(n) = u(n–1) + 2$, $u(1) = 9$ b) $u(n) = 3u(n–1)$, $u(1) = 3$ c) $2u(n–1) –1$, $u(1) = 3$